PHILOSOPHIA MATHEMATICA

PHILOSOPHY OF MATHEMATICS,
ITS LEARNING, AND ITS APPLICATION

PHILOSOPHIE DES MATHÉMATIQUES,
LEURS ÉTUDE ET APPLICATIONS

Offprint from Volume 7 (1999), Number 1

Critical Study / Book Review

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THE PHILOSOPHICAL JOURNAL OF
THE CANADIAN SOCIETY FOR HISTORY
AND PHILOSOPHY OF MATHEMATICS

LA SOCIÉTÉ CANADIENNE D'HISTOIRE ET
DE PHILOSOPHIE DES MATHÉMATIQUES

SERIES III

MICHAEL RESNIK. ***Mathematics as a Science of Patterns.*** Oxford: Oxford University Press, 1997. Pp. xiii + 281. ISBN 0-19-823608-5.

Reviewed by MARK BALAGUER*

In this interesting and important book, Michael Resnik brings together various ideas on the philosophy of mathematics that he has been developing over the last three decades. I highly recommend the work to everyone. Those with only a passing knowledge of Resnik's work will perhaps be surprised to learn that there is much more to his philosophical view than the structuralism for which he is so well known. And even those who are more familiar with Resnik's work will profit from this book, for they will see how Resnik weaves together the various theses that he has argued for over the years into one overall view. I will try to characterize this overall view in section 1. I will begin by laying out the individual theses that Resnik defends, and then I will explain how they all hang together in Resnik's system. Then in section 2, I will critically discuss various facets of Resnik's view.

1. A Characterization of Resnik's Program

The most important theses that Resnik defends are as follows.

1. Mathematical realism (*i.e.*, the thesis that mathematics is a science of abstract mathematical objects that exist objectively and independently of us, outside of space and time).

2. Mathematical structuralism (*i.e.*, the thesis that mathematics is a science of abstract mathematical structures, or patterns, and that mathematical objects are just positions in such patterns).

3. Ontological relativity. (On Resnik's view, this is the thesis that in certain contexts, there are no facts of the matter as to the answers to various questions about the natures and identities of certain objects. Resnik is concerned mainly with ontological relativity about mathematical objects, or what he calls the *incompleteness of mathematical objects*. This is the thesis that there are no facts of the matter as to the answers to certain questions about the natures and identities of mathematical objects, *e.g.*, questions like 'Are numbers sets?' and 'Is $2 = \{\{\emptyset\}\}$?')

4. Indispensability (*i.e.*, the thesis that mathematics is indispensable to natural science).

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5. Confirmation holism (*i.e.*, the thesis that the 'evidence for a scientific theory bears upon the theoretical apparatus as a whole rather than upon individual... hypotheses' (p. 45)).

6. Empiricism about mathematics (and everything else for that matter).

7. Anti-separatism about mathematics and science (*i.e.*, the thesis that there is no clear distinction between mathematics and natural science).

8. Blurriism about the abstract-concrete distinction.

9. Naturalism (*i.e.*, the thesis that natural science is the ultimate arbiter of truth and existence).

10. Disquotationalism about truth and reference.

11. Postulationalism (*i.e.*, the thesis that abstract mathematical objects are epistemic posits).

12. Definitionism about mathematical axiom systems (*i.e.*, the thesis that axiom systems form implicit definitions of the mathematical structures that the axiom systems are about). (Resnik notes that by taking this line, he does not mean to be suggesting that axiom systems are 'known a *priori*'(p. 237).)

13. Logical non-cognitivism. (This is a form of anti-realism about logic according to which there are no such things as logical properties or relations, and so sentences of the form 'P entails Q', 'P is inconsistent', and so on are non-factual and, hence, never true.)

Resnik's book is an extended argument for, and defense of, mathematical realism (thesis 1). More specifically, Resnik mounts one positive argument in favor of realism and tries to block four different objections to realism. All of the theses listed above aid in this project in one way or another.

The positive argument for realism is based on indispensability (thesis 4), but Resnik's argument is stronger and more sophisticated than most indispensability arguments in the literature. In particular, Resnik intends his argument to apply to all uses of mathematics in science, even when we do not think the given scientific theory or hypothesis is literally true, and he intends it to apply to all of mathematics, even those parts that have never been applied. Resnik begins by stating a fairly standard indispensability argument but then moves on to what he calls the 'pragmatic indispensability argument' (p. 46). In a nutshell, this argument proceeds as follows: the descriptions and inferences of science indispensably assume the existence of many mathematical objects and the truth of many mathematical theories; but if we combine this with naturalism (thesis 9), we obtain the result that this underlying presupposition of science needs to be taken seriously, *i.e.*, that we cannot just claim that scientists are confused here; but this seems to suggest that we ought to countenance the existence of many mathematical objects and the truth of many mathematical theories. (This argument does not assume that the science in question is literally true: even if it isn't, we are still committed to the mathematics, Resnik thinks, because we could not do science at all without mathematics.) Now, this pragmatic argument only provides a reason to endorse the parts of mathematics that get applied in science. Resnik's argument for the claim that we should be realists about the whole of mathematics is complex, but in brief, the idea is that when we combine the pragmatic argument with holism (thesis 5), we are led to conclude that mathematics is a trustworthy discipline and, hence, that we ought to accept its findings and the sorts of justificatory methods that mathematicians use, *e.g.*, proofs.

The justificatory methods of mathematicians deliver what Resnik calls *local* sources of evidence, which are to be contrasted with the *global* sources of evidence that are relevant to the pragmatic indispensability argument. The most obvious sort of local mathematical evidence is the sort we get from mathematical proofs, but it is important to note that, for Resnik, this evidence is not *a priori*. He uses his holism to argue that all evidence, including proof-based evidence, is empirical. Thus, empiricism (thesis 6) is also relevant to Resnik's argument for realism. And also relevant here are anti-separatism about mathematics and science (thesis 7) and blurri-ism about the abstract-concrete distinction (thesis 8). They are central to (a) Resnik's version of holism, (b) his move from the global evidence that the pragmatic argument delivers to the local evidence that mathematical methods deliver, and (c) his argument for the claim that this local evidence is empirical. Finally, logical anti-realism (thesis 13) is also relevant to (c): one might think that the knowledge that arises out of local mathematical evidence is *a priori* because it is based on logical deduction; but part of Resnik's response to this is that logical deductions do not tell us that certain sentences have a special metaphysical property of logical necessity, because there is really no such property.

The first objection to realism that Resnik considers, which we can call the semantic objection, is that (a) realists seem committed to a correspondence theory of truth and (b) the correspondence theory is implausible. Resnik's response is very simple: he rejects the correspondence theory of truth and shows how realists can endorse a disquotational theory of truth (thesis 10).

The second objection to realism that Resnik considers might be called the multiple-reductions objection, or the Benacerraf [1965] objection. This objection has been formulated in a number of different ways, so it might be wise to say that there is actually a cluster of related objections to realism here, but in any event, Resnik formulates the problem like this: (a) realism seems committed to there being facts of the matter as to the answers to questions like 'Are numbers sets?' and 'Is 2 identical with $\{\{\emptyset\}\}$?'; but (b) if we look at mathematical practice and our mathematical theories, it seems that there are no facts of the matter about such things. Resnik's response to this objection is similar to his response to the first objection; he accepts the claim that there are no facts of the matter here, and he explains how this is consistent with realism. In making his case, Resnik appeals to structuralism and ontological relativity (theses 2 and 3). According to Resnik's version of realism, mathematical objects are just positions in patterns, and these positions are incomplete with respect to the properties that they possess; more specifically, they have no distinguishing properties other than those that they have in virtue of the relations that they bear to other positions in the same pattern. Thus, according to this version of realism, there *really aren't* any facts of the matter about the answers to questions like 'Are numbers sets?' and 'Is $2 = \{\{\emptyset\}\}$?'

The third objection to realism—the epistemological objection, or the Benacerraf [1973] objection—is that realism seems incompatible with the fact that we (*i.e.*, spatio-temporal human beings) have mathematical knowledge. Resnik spends more time trying to block this objection than he spends on anything else. Indeed, I think it is fair to say that at least half the book bears upon this objection to platonism. It seems to me that there are several different ideas at work here. As a sort of preliminary point, Resnik appeals to anti-separatism and blurriism (theses 7 and 8) in order to motivate the claim that knowledge of various kinds of concrete objects is just as problematic as knowledge of abstract objects. His own positive epistemology of mathematical objects begins with an explanation of how we might have come to have theories and beliefs about such objects in the first place. In this connection, Resnik provides a quasi-historical-quasi-speculative sketch of how we might have come to endorse the sorts of mathematical beliefs that we do. Central to the discussion are an appeal to postulationalism (thesis 11) and our capacity for pattern cognition. What Resnik tries to explain here is how, in our dealings with structured systems of physical objects, we might have slowly come to posit abstract mathematical structures and develop theories of them.

But this still leaves the problem of explaining how we could have *knowledge* in connection with our mathematical theories and beliefs. In this connection, Resnik pursues three different lines of thought. The first is based most centrally on holism. Resnik does not put it precisely this way, but as I read him—and he has approved of this reading in correspondence—the idea behind this first strategy of accounting for knowledge of abstract objects is simply to appeal back to all the considerations that motivate realism. The reason this makes sense is that (a) Resnik tried to motivate realism by simply explaining how we do have good reasons for believing that there exist abstract mathematical objects (and that our mathematical theories accurately characterize them), and (b) explaining how we have good reasons for these beliefs is just to explain how these beliefs are justified and, hence, how we have knowledge here. The second strategy that

Resnik pursues in attempting to account for mathematical knowledge is based on an appeal to definitionism (thesis 12). The idea here is that our axiom systems yield knowledge of mathematical structures, as opposed to mere beliefs about such structures, because they provide implicit definitions of these structures. In other words, we can know that our axiom systems provide true descriptions of the structures that they are about, because they, so to speak, cannot *help* but be true, because they are definitions. Resnik's third and final strategy is to appeal to structuralism (thesis 2) in an attempt to motivate the idea that we acquire mathematical knowledge via pattern cognition. His claim here seems to be that we can acquire knowledge of mathematical structures via perceptual acquaintance with systems of physical objects that instantiate (or partially instantiate) these structures.

The fourth objection to realism—the reference objection—is this: realism entails that we can refer to abstract objects, but this is impossible, because (a) such objects are causally inert, and (b) reference involves causal processes. This objection is very similar to the third objection, and so I think that Resnik would say that much of what he says in response to the third objection applies to this fourth objection as well. But in addition to this, Resnik makes a separate point here: he endorses a disquotational theory of reference (thesis 10), and so he rejects causal theories of reference, and so there is simply no reason to think that the causal inertness of mathematical objects raises any obstacle to our ability to refer to them.

So that is a quick sketch of Resnik's overall program. Now, there is more to this book than these remarks suggest. For instance, in addition to motivating and defending his own view, Resnik also provides some interesting criticisms during the course of the book of various alternative views that have been proposed by recent philosophers of mathematics, *e.g.*, Field, Chihara, Hellman, Kitcher, and the early Maddy. And I have also said nothing here about the way in which Resnik motivates and defends the various individual theses with which I began this section. Now, some of his arguments here will emerge in the next section, as I discuss various facets of Resnik's view. But for now, the above sketch will have to suffice.

2. Critical Remarks

As I have said, I think that Resnik's book is an extremely rich and interesting one. Moreover, the book is loaded with very bold claims, backed up by interesting arguments and explanations. Finally, many of Resnik's theses are highly controversial, and while my own view is similar to his in some ways, it is also very much opposed to his view in other ways. Thus, there is a lot that I would like to say about this book, and I cannot even come close to saying it all here. What follows are merely a few scattered remarks.

2.1 Indispensability, Holism, and the Case for Realism

I begin with a few words about Resnik's positive argument for realism. I do not think this argument succeeds because I think it relies upon a radical sort of holism that is false. And I should note that this is not just an objection to Resnik's way of formulating the indispensability argument. I think that all indispensability arguments rely upon a radical sort of holism and that when we appreciate this, and why this holism fails, we see that the whole idea of using indispensability to argue for mathematical realism is doomed to failure. Let me say a few words about this.

It may be that some sort of holism is true in connection with the nominalistic aspects of our empirical theories. I shall not discuss this here, because it is not relevant to the problem that I have with indispensability arguments. The problem I have is with versions of holism that claim to hold up across the divide between the empirical and the concrete on the one hand and the mathematical and the abstract on the other. It seems to me that even if our mathematical theories are indispensable to the descriptions and inferences of empirical science, this gives us no reason whatsoever to believe that these mathematical theories are literally true or that there are any such things as abstract mathematical objects. The reason, in a nutshell, is that abstract objects are supposed to be causally inert. We can think of it, metaphorically, like this: if all abstract objects suddenly disappeared, the physical world would remain unchanged. But this suggests that if there were never any such things as abstract objects to begin with, the physical world would be exactly as it is right now, and we would be receiving the very same empirical data that we are presently receiving. Indeed, insofar as the realist view takes abstract objects to be causally inert, it *predicts* that if there were no mathematical objects, the physical world would be as it is right now. Thus, it seems to me that whatever empirical data we are receiving right now, we cannot take these data to support the existence of abstract objects, for even if there were no such things as abstract objects, we would still be receiving these same data. And it should be noted that this objection applies not just to Resnik's use of holism to try to extend his argument to the unapplied portions of mathematics, but to the first part of his argument as well, *i.e.*, to what he calls the pragmatic indispensability argument. To put the point in Resnik's own terms, even if science does assume the existence of abstract objects, this gives us no reason to believe in such objects, because even if there were no such things as abstract objects, science could be practiced exactly as it is right now (assumptions of *abstracta* and all) with exactly the same results. In short, even if science presupposes abstract objects, we can rationally endorse a useful-fiction stance toward this presupposition.

How would Resnik respond to this? Well, first of all, I should note that he is well aware that people might try to attack his view in this way: he says that his 'holism would be unacceptable if... one could establish an epistemically principled division between the empirical and formal sciences' (p. 135). What I am suggesting is that there is a principled epistemic division here because there is a principled ontological division. But, of course, we know what Resnik would say to this—he denies that there is any principled ontological division here because he is a blurriist about the abstract-concrete distinction. It seems to me, however, that blurriism is one of the more implausible theses that Resnik defends in his book. What I find puzzling in Resnik's view is not so much his blurriism as the fact that he embraces blurriism together with the thesis that abstract objects exist 'outside space and time' (p. 82). For it seems to me that the thesis of non-spatio-temporality brings with it a very clear abstract-concrete distinction. Thus, if Resnik rejected the non-spatio-temporality of abstract objects, then I would find his blurriism (and hence, his holism) more palatable. But he does not do this. What he does instead is call into question the spatio-temporality—or perhaps more accurately, the *full-blown* spatio-temporality—of physical objects. Resnik's argument here is based on the claim that quantum systems often lack spatio-temporal locations. More specifically, his claim is that if a quantum system is in a superposition of position states, then it has no spatio-temporal location.

This argument seems misguided to me. In the first place, it is not obvious that when a quantum system is in a superposition of position states, we ought to say that it has no spatio-temporal location; it might be better to say that it does have a location but that its location state is a superposition state. Second, and more important, even if we do say that quantum systems in such states have no spatio-temporal locations, it does not seem accurate to say that they exist outside of space and time in the way that abstract objects do, because they still have other spatio-temporal properties. Moreover, we can always, so to speak, 'make' a quantum system have a determinate spatio-temporal location by simply measuring its position. Thus, quantum systems are not abstract objects—or even borderline cases, as Resnik suggests—because abstract objects *never* exist in spacetime and *could not* exist in spacetime. This seems to me to provide us with a perfectly principled ontological distinction, and what's more, it seems very much in line with standard usage: if an object could *ever* have a *single* spatio-temporal property, then it is not abstract; if it can enter into *any* causal relations, then it is not abstract; and so on.

Another argument that Resnik offers for blurriism is that under certain descriptions, quantum fields 'are little different from functions from spacetime to probabilities' (p. 104). Now, one obvious response that an anti-blurriist could make here is that while quantum fields can be *represented* by probability functions, they are not themselves identical with such things—*i.e.*, they are not mathematical objects. Resnik claims that this response is available only to anti-realists about quantum fields and superposition states, but I do not see why he says this—I do not see why a full-blown realist about quantum fields and superposition states cannot maintain that while fields can be represented by probability functions, they are not functions themselves.

If holism and the indispensability argument depend upon blurriism, then it seems to me that they are in serious trouble. Now, I think that most proponents of holism and the indispensability argument would deny any dependence on blurriism, but I do not see how such a view could be made out, for again, it seems to me that standard versions of anti-blurriism entail that abstract objects are causally inert, and I think this deals a serious blow to both holism and the indispensability argument.

I cannot say anything more about the indispensability argument here. This is an extremely complicated issue, and to say anything of real philosophical value would take a lot more space than I can dedicate to it here. All I can say is this: (a) I think Resnik is right that this argument rests upon a version of holism that cuts across the abstract-concrete divide; but (b) I think that all such versions of holism are false. But I cannot argue for (b) here; the above remarks provide merely a hand-waving beginning of such an argument.[1] (I should note here that Resnik discusses this issue at length. He is aware that a lot of philosophers of science and mathematics would endorse (b), and large chunks of his book are dedicated to responding to such arguments and motivating his holism. I am not convinced by his arguments, but I do think they are interesting and important and worth reading.)

2.2 Structuralism and the Multiple-Reductions Objection

As we saw above, Resnik formulates the multiple-reductions objection to realism as follows: (a) realism seems committed to there being facts of the matter as to the answers to questions like 'Are numbers sets?' and 'Is 2 identical with $\{\{\emptyset\}\}$?'; but (b) if we look at mathematical practice and our mathematical theories, it seems that there are no facts of the matter about such things. Resnik does a nice job, I think, of showing that (a) is false by explaining how realists can avoid facts of the matter here by endorsing structuralism and ontological relativity. But I have a few reservations.

First, I do not approve of Resnik's formulation of the multiple-reductions problem, because I doubt that (b) is true. I grant that there may be *some* mathematical questions about which there are no facts of the matter, but I do not think this is the case with questions like 'Are numbers sets?' and 'Is $2 = \{\{\emptyset\}\}$?' The only argument Resnik has for (b) is that nothing in mathematical or scientific theory entails an answer to such questions. But nothing in mathematical or scientific theory entails an answer to the ques-

[1] I give a fuller version of the argument in my [1998], chapter 7.

tion of whether 2 is identical with Stalin's paranoia, but despite this, we are not inclined to conclude that there is no fact of the matter here. Now, in saying this, I do not mean to be questioning Resnik's structuralism or ontological relativity. For we can admit that mathematical objects like 2 and $\{\{\emptyset\}\}$ are incomplete with respect to the properties that they possess but deny that they are incomplete with respect to the question 'Is $2 = \{\{\emptyset\}\}$?' My point is simply that we should not conclude that they are incomplete on this score simply because our axiomatic mathematical theories do not settle the matter. If we look at mathematical practice *as a whole*, it is, I think, apparent that there are very definite facts of the matter about these questions: mathematicians think that numbers are *not* identical with sets and that 2 is not identical with $\{\{\emptyset\}\}$.

I think the proper way to formulate the multiple-reductions objection is as follows: (a) one of the central ideas behind realism is supposed to be that our mathematical theories are about *unique* collections of abstract objects—*e.g.*, arithmetic is supposed to be about *the* natural-number sequence—but (b) when we look closely, it seems that realism is, in fact, incapable of delivering this result. This strikes me as important because I do not think it is at all clear that we can solve this problem by merely appealing to structuralism and ontological relativity. Now, *prima facie*, it might seem that we can. More specifically, it might seem that structuralists can solve the above problem by saying something like this:

> There are an infinite number of sequences of objects in the mathematical realm that one might take to be the natural-number sequence and that seem to be perfectly good candidates for being the natural-number sequence, but arithmetic is not about any particular one of these sequences. Rather, it is about the structure that they all have in common.

But I do not think this really solves the problem. According to the sort of realistic structuralism that we are discussing here, structures exist independently of us and outside of spacetime. (This is clearly Resnik's view; see, *e.g.*, pp. 4 and 82.) Thus, even if mathematical objects are just positions in structures, and even if they are incomplete with respect to the properties they possess, it still seems to me very likely that there are multiple structures that exist outside space and time, that satisfy all of the *desiderata* for being the natural-number sequence, and that differ from one another only in ways that no human being has ever thought about. Thus, I do not think it is at all obvious that structuralists can salvage unique structures for our mathematical theories to be about, and so I think there is a worry here that Resnik needs to address.

Now, if Resnik were to endorse what might be called *radical ontological relativity* about positions in patterns—*i.e.*, the thesis that positions in patterns do not have any non-structural properties (*i.e.*, that they have no properties except for those that they have in virtue of the relations they bear to other positions in the same structure)—then he might be able to avoid my version of the multiple-reductions objection. For given radical ontological relativity, one might argue that (a) any two structures that are isomorphic are identical (*i.e.*, not really two but one),[2] and (b) our mathematical theories (or at least some of them) are about unique structures up to isomorphism. Now, at times, it does seem that Resnik would endorse this sort of view—*e.g.*, he says that positions 'have no identity or distinguishing features outside a structure' (p. 201)—but he has assured me in correspondence that he did not mean to be endorsing radical ontological relativity here. He is not saying that positions have no non-structural features; he is saying that they have no *distinguishing* non-structural features. What this means will become clear shortly, but first, it is worth noting that Resnik was wise to avoid radical ontological relativity, because that view is untenable. That the view is false can be appreciated by merely noting that positions are non-spatio-temporal and that this is a non-structural property. But it gets worse. For the property of having only structural properties is itself a non-structural property, and so it would seem that radical ontological relativity is incoherent.

But again, Resnik's version of structuralism does not entail that positions have no non-structural properties. It entails only that they have no *distinguishing* non-structural properties. In correspondence, Resnik says that what he means by this is that positions 'have no non-structural properties sufficient to distinguish... [them] from all other positions'.[3] This view does not fall prey to the easy refutation that radical ontological relativity falls to. But neither does it deliver uniqueness (and recall that the desire to capture uniqueness was what led us to radical ontological relativity in the first place). For given that structures exist independently of us in an abstract mathematical realm, it seems that Resnik has to allow that there may be multiple structures that satisfy all of the *desiderata* for being the natural-number sequence and differ from one another only in ways that no human being has ever imagined; for structures of this sort could differ in the non-distinguishing, non-structural properties that their positions possess. (Properties could serve to distinguish two positions, and hence two structures, and still count as non-distinguishing properties, because they could fail to distinguish the positions in question from *all* other positions.)

Perhaps Resnik could avoid this problem by strengthening his view and claiming that the only non-structural properties that positions possess are

[2] I should note that Resnik clearly rejects thesis (a); so I am not talking about his view here.

[3] Given this definition of 'distinguishing property', it seems to me that Resnik could capture his view by simply saying that positions have no distinguishing properties. He does not need to say that they have no distinguishing non-structural properties because their structural properties will not distinguish them from all other positions (although they will distinguish them from all other positions in the same structure).

properties that *all* positions possess. Or perhaps there is a way to weaken this somewhat and still obtain uniqueness. Perhaps. But perhaps not. I have not thought about this long enough to have a strong opinion about it. But it does seem to me fair to say that if Resnik wants to salvage uniqueness, then he needs to (a) provide a theory that tells us the exact kinds of properties that positions can possess and (b) explain how this theory does in fact deliver uniqueness. It seems to me far from obvious that this could be done (and less obvious that it could be done in a way that preserved the structuralist intuition that mathematical objects have only the sorts of properties that bare *positions* have).

I should point out, however, that Resnik does not *need* to salvage uniqueness. Indeed, there are two other strategies he could pursue here, in addition to the strategy of trying to salvage uniqueness. One strategy would be to try to *abandon* uniqueness; the idea here would be to (a) admit that there *are* multiple structures that are distinct and that satisfy all of the *desiderata* for being the natural-number sequence, and (b) try to explain why this fact is consistent with realism. And a second strategy would be to try to claim that there is *no fact of the matter* about uniqueness, *i.e.*, no fact of the matter as to whether, *e.g.*, there is a unique structure that is the natural-number sequence. In some ways, this last strategy seems to fit best with other things Resnik says, *e.g.*, that there is no fact of the matter as to whether $2 = \{\{\emptyset\}\}$. But in any event, let me say a few words about these two alternative strategies.

The main point that needs to be made about the no-fact-of-the-matter strategy is that it faces the same problem that the strategy of trying to *salvage* uniqueness faces. More specifically, the no-fact-of-the-matter strategy is problematic because it is not clear that there is a coherent view of the sorts of properties that positions can possess that would deliver the desired result that there is no fact of the matter as to whether there is a unique natural-number sequence. Resnik's present view of distinguishing properties clearly does not deliver this result because it does not rule out the possibility of there being multiple structures that are clearly distinct (because their positions have different non-distinguishing non-structural properties) and that have equally strong claims to being the natural-number sequence.

What about the strategy of trying to *abandon* uniqueness? Well, my own view is that this is the *best* strategy for realists to pursue, but I am not going to argue this here. I would like to point out, however, that *prima facie*, structuralism seems entirely irrelevant to this strategy; that is, it seems that if structuralistic realists can embrace non-uniqueness and explain why it is consistent with realism, then object-platonists can too. If this is right, and if I am also right that this is the strategy that realists ought to pursue, then it follows that structuralism is irrelevant to the multiple-reductions objection. But I have not argued these points here.[4] All I have argued here is that (a) it is not obvious whether structuralism can provide realists with a means of solving the multiple-reductions problem, and (b) in order to show that it can, Resnik would need to provide a precise account of the sorts of properties that positions can possess and then argue that this account delivers the desired uniqueness result (or no-fact-of-the-matter-about-uniqueness result).

Before going on, I should note that the question I am raising here about the sorts of properties that positions can possess seems to me to strike at the very heart of structuralism. Above, I said that structuralism is the view that our mathematical theories are descriptions of abstract mathematical structures, or patterns, and that mathematical objects are just positions in such patterns. In response to this, however, one might inquire after the difference between positions and traditionally conceived mathematical objects (and between structures and traditionally conceived systems of mathematical objects). What Resnik would say to this is that positions are different from traditional mathematical objects because they are incomplete. Thus, it seems to me that structuralism more or less *reduces* to the thesis that mathematical objects are incomplete with respect to the properties they possess (or as Resnik would put it, with respect to the facts that obtain about their natures and identities). I want to make three points about this. First, it now seems that structuralism (thesis 2) is just a special case of ontological relativity (thesis 3); for we have already seen that the thesis of incompleteness is a special case of ontological relativity. Second, given that structuralism reduces to the thesis of incompleteness, the question I was asking above about the sorts of properties that positions can possess seems to be a question about what exactly structuralism *says*. For in asking what sorts of properties positions can possess, we are really just asking structuralists to tell us the *degree* to which mathematical objects are incomplete. And third, it is not obvious that the appeal to incompleteness provides structuralists with a genuine difference between positions and traditional mathematical objects; for it is not obvious that object-platonists are committed to the thesis that mathematical objects are complete, in Resnik's sense of the term.

2.3 The Epistemological Objection

Resnik's non-causal postulational view of how we come to have beliefs and theories about mathematical objects strikes me as quite plausible. I do not agree with everything he says in his quasi-historical sketch, but the central idea here—that realists can account for our having beliefs about non-spatio-temporal objects by adopting a sort of postulationalism—seems right to me. When we come to Resnik's explanation of how we could have

[4] I argue these points in my [1998], chapter 4.

knowledge of mathematical objects, however, I have some worries. As I said above, Resnik pursues three distinct lines of thought here. Now, I have already commented on the first, for recall that it relies upon Resnik's positive argument for realism, and I have already commented on that argument. (More specifically, Resnik's first strategy here rests heavily upon his holism, and I have already said why I think his holism is untenable.) But what I would like to do now is say a few words about the second and third strategies that Resnik pursues in trying to explain how we could have knowledge of mathematical objects.

Resnik's second strategy is, I think, the best of the three. The central premise behind this strategy is definitionism (thesis 12). The idea is that our axiomatic mathematical theories provide us with knowledge of mathematical structures because they provide implicit definitions of such structures. I think that Resnik is on the right track here. Indeed, it seems to me that the only real problem with this response to the epistemological objection is that there is an implicit premise here that Resnik needs to (a) make explicit and (b) defend. To see this, consider the following objection that one might raise to Resnik's implicit-definition strategy.

> Look, I see the point you're trying to make here: we know that the various theorems of our mathematical theories are true because we can prove them from the axioms of these theories; and to ask how we know that the axioms are true is confused, because they are, in some sense, *stipulative*, because all they do is specify the objects that we're speaking of. But there is an obvious response to this: it is, of course, true that once we lay down some definitions, we can derive consequences from them; but we cannot acquire any factual knowledge by merely laying down definitions, because in merely specifying a definition, it doesn't follow that there is anything in the world that answers to the definition. Thus, for instance, suppose that I stipulate that 'Broctoon' is to refer to the fourth red-headed daughter of L. Ron Hubbard. I can deduce from this that Broctoon is female. But this does not constitute real knowledge unless I first know that L. Ron Hubbard really does have a fourth red-headed daughter. Without this, my so-called knowledge is vacuous. Likewise, you can lay down some axioms that define a mathematical structure and then prove some theorems about the nature of this structure, but unless you tell us some story about how human beings could know *which* of our mathematical axiom systems actually pick out systems of mathematical objects, you will not have solved the epistemological problem with realism.

The first point to note here is that this objection should not be taken as requesting an explanation of how we could know that there are any such things as mathematical objects. Such an objection would not be legitimate, and indeed, Resnik points this out himself. But the above objection is no more asking for an explanation of how we could know that there are any such things as mathematical objects than the objector in the Broctoon case is asking for an explanation of how we could know that there is any such thing as an external physical world. This objector might say: 'Look, I'll *grant* you that there's an external world, but before I can grant you that there's any real knowledge in the Broctoon case, you need to tell us how the alleged knower knows that there's any such person as Broctoon.' Likewise, those who think there is an epistemological problem with realism might say: 'Look, I'll grant you that there are mathematical objects, but before I can grant that some *particular* axiom system gives us any real knowledge of the mathematical realm, you need to tell us how we could know that the particular objects that this system is supposed to be about actually exist.'

The response that Resnik ought to give here, in my opinion, is that the mathematical case is not analogous to the Broctoon case because in mathematics, *every* consistent (purely mathematical) axiom system characterizes a structure, because the mathematical realm is plenitudinous, *i.e.*, because all the structures that possibly could exist actually do exist. This is the implicit assumption that I mentioned above. Resnik needs to say that ontological plenitudinousness is built into structuralistic mathematical realism; for given this, he can say that by laying down definitions and axioms in mathematics, we are merely indicating *which* objects we are talking about. Thus, on this way of proceeding, the sort of problem that arises in the Broctoon case would not arise at all in mathematical cases. Now, of course, taking this stance would require quite a bit of defense, but I for one think that a successful defense can be made,[5] and in any event, I think that the implicit-definition strategy can *only* work if it is combined with an appeal to plenitudinousness, and so I think that Resnik should have explicitly introduced and defended this idea.

I now move on to Resnik's third strategy for addressing the epistemological objection to realism. This strategy features an appeal to our capacity for pattern cognition (and since structuralism is apparently what makes talk of patterns relevant here in the first place, it involves an appeal to structuralism as well). Now, I will argue shortly that there is nothing epistemically important in Resnik's appeal to pattern cognition that is not already present in his appeal to definitionism. But first, I would like to argue that structuralism itself is entirely irrelevant to the appeal to pattern cognition. The reason is that (a) traditional object-platonists can maintain that all of the mathematically important properties of mathematical objects are structural properties—*i.e.*, properties they have in virtue of the relations they bear to other mathematical objects—and, therefore, (b) if structuralists can claim that some of our mathematical knowledge arises out of pattern cognition, then object-platonists can too. In other words, you do not have to claim that mathematical objects are incomplete, in Resnik's sense of the term, in order to maintain that structural facts are

[5] Indeed, I provide such a defense in my [1998], chapter 3.

what are important in mathematics or that, because of this, pattern cognition is important to the epistemology of mathematics. But as far as I know, the only reason realists have appealed to structuralism in connection with the epistemological problem has been to enable them to appeal to pattern cognition. Thus, since structuralism really is not needed here, it seems that it is simply irrelevant to the epistemological problem with realism. (And note that if we combine this with the point I argued in section 2.2—that it is not obvious that realists can use an appeal to structuralism to solve the *multiple-reductions* problem—it becomes unclear whether structuralism does anything philosophically important for realists. For the traditional advertisement for structuralism is that it provides realists with a means of solving the multiple-reductions problem and the epistemological problem. I do not know what else it is supposed to do.)

In any event, I now want to argue that there is nothing epistemically important in Resnik's third strategy (*i.e.*, his appeal to pattern cognition) that is not already present in his second strategy (*i.e.*, his appeal to definitionism). Resnik's central claim regarding the epistemic import of pattern cognition is that 'systems of physical objects instantiating... patterns can inform us of properties of mathematical objects' (p. 224). As I read this, the claim here is that we can acquire knowledge of abstract patterns via perceptual contact with systems of concrete objects that instantiate (or partially instantiate) these patterns. But to this, one might respond as follows.

> Human beings could not learn anything about any abstract pattern by perceiving a system of concrete objects unless they knew in advance that the given abstract system stood in some particular relation to the given system of concrete objects. But how could human beings ever know this? Since they have no epistemic access to abstract patterns, it seems that they could not.

Resnik would reply to this by pointing out that abstract patterns are *posits* and that what we are positing here are precisely patterns that *do* bear appropriate structural relations to certain systems of physical objects. (See, for instance, page 229.)

This is a nice point. But now it seems to me that Resnik's appeal to pattern cognition has collapsed into his appeal to definitionism. For his point here is that it is confused to ask how we know that mathematical structures are appropriately related to the systems of physical objects that we abstract away from in positing these structures, because this is *stipulative*—*i.e.*, it is built into the act of positing that we are positing structures of the appropriate sort. Thus, I do not think there is anything here that is not already contained in the appeal to definitionism. And because of this, I do not think the notion of pattern cognition is doing any important epistemological work here. (Pattern cognition may well be relevant to a theory of how we come to our mathematical beliefs and theories in the first place, as opposed to our mathematical knowledge—Resnik certainly thinks it is relevant here, and I think he is probably right—but it should be noted that even if this is right, structuralism itself will not be relevant here, because, as we have seen, traditional object-platonists can appeal to pattern cognition as easily as structuralists can.)

2.4 Disquotationalism, the Semantic Objection, and the Reference Objection

Resnik endorses disquotationalism in an effort to block two different objections to realism. The first (the semantic objection) is that realists seem committed to a correspondence theory of truth that seems implausible. And the second (the reference objection) is that our best theories of reference seem to be causal theories that are incompatible with mathematical realism. Resnik's response to the first problem is that realists are not committed to the correspondence theory of truth because they can endorse disquotationalism. This is certainly right: disquotationalism is undoubtedly consistent with realism. But I think that Resnik overstates the whole problem here and, by endorsing disquotationalism, he overreacts. It seems to me that disquotationalism is a radical, counterintuitive theory, and that there are no serious problems with the correspondence theory that motivate a shift to this sort of theory. Part of the problem here is that Resnik takes the correspondence theory to be a much stronger theory than it actually is. Once we have a disquotational notion of truth-in-L, the correspondence theory can be developed by merely introducing a notion of a speaker *intending* an utterance to be taken as a token of a sentence in a given language. Thus, to put the point very roughly and imprecisely, a sentence token is *true*, on this view, if and only if (a) it was intended to be an L-token, and (b) it (or more precisely, the corresponding sentence type) is disquotationally true-in-L.[6]

Of course, various philosophers, Resnik among them, have taken the correspondence theory to entail more than this. For instance, some people have suggested that it entails a causal theory of reference. Resnik stops short of this, but he does suggest that the correspondence theory entails that there is a specifiable word-world reference relation that applies to arbitrary languages. But as far as I can see, we can endorse a correspondence theory of truth without saying much of anything about reference. Again, we can develop the correspondence theory by combining it with a theory of the intention relation that obtains between speakers and languages (or a theory of when a language is the language of a given population of speakers) rather than with a causal theory of reference. Now, it is of course true that two of the most prominent advocates of the correspondence theory in recent years, namely, Field and Devitt,[7] have also endorsed causal theories

[6] For more on views of this sort, see Soames [1984] and Field [1986].

[7] See Field [1972] and Devitt [1984].

of reference. But this doesn't mean anything—especially when we consider that both Field and Devitt are anti-realists about abstract objects.

In any event, the fact that correspondence theorists need not endorse a causal theory of reference also suggests that mathematical realists need not endorse disquotationalism in order to avoid the reference objection. I agree with Resnik that realists should respond to this objection by rejecting the causal theory of reference, but they do not need to endorse disquotationalism in order to do this, and *contra* Resnik, I do not think they should.

I said above that disquotationalism is counterintuitive. The reason I say this is that I take that theory to entail that, *e.g.*,

(1) All satellites follow elliptical orbits

is equivalent to

(2) 'All satellites follow elliptical orbits' is true;

but *prima facie*, it seems that (1) and (2) are not equivalent because, intuitively, they need not have the same truth value. Consider possible worlds in which (1) is true and, moreover, in which there are no speakers or abstract objects. In such worlds, there are no such things as sentences (tokens or types), and so it would seem that (2) is not true in such worlds. (I am assuming here that the quoted sentence in (2) is a singular term that denotes a sentence; of course, disquotationalists could reject this assumption, but they would need to motivate this, for intuitively, the assumption seems right.)

2.5 Anti-Realism about Logic

Resnik is an empiricist about mathematics, and so he wants to block the worry that mathematical knowledge is *a priori* because it is based largely on proof and logical deduction. His position here is that logical deductions do not tell us that certain claims are logically necessary, because there is no such property as logical necessity. More specifically, Resnik denies that sentences of the form 'P entails Q' are ever true. Such sentences can be used, according to Resnik, to guide our inferential practices, or to indicate that we endorse certain other sentences that are factual—*e.g.*, by saying 'P entails Q', we indicate that we endorse 'if P then Q'—but sentences like 'P entails Q' are not themselves factual. Thus, Resnik endorses anti-realism about logic. He calls his view *logical non-cognitivism*.

Now, *prima facie*, this seems a bit puzzling. For in the case of mathematics, Resnik's empiricism does not lead him to endorse anti-realism. He believes that the only ultimate sources of mathematical knowledge are empirical, but he still maintains that mathematical sentences are about abstract objects and that they are true or false independently of us and our theorizing. Thus, one might wonder why Resnik endorses a radical sort of logical anti-realism, as opposed to some version of logical empiricism—*i.e.*, as opposed to a view that takes an empiricist stance on logical knowledge but allows that certain sentences *really are* logically true and that certain sentences *really do* entail certain other sentences. In short, one might wonder why Resnik adopts such different stances with respect to mathematics and logic. And one might guess that the answer would be that Resnik thinks there are good independent arguments in favor of logical anti-realism. But surprisingly, Resnik says almost nothing in this connection. He spends a great deal of time attacking various arguments in favor of realism and against anti-realism, but as far as positive arguments for anti-realism (or against realism) are concerned, I find only one argument, an epistemological objection to realism whose entire content is contained in less than one sentence. He says:

> if logical necessity is a metaphysical property of sentences or propositions, then we have no grounds for thinking that we can always know where it applies (p. 162).

I have a hard time believing that Resnik thinks this little argument bears so much weight, especially since he says nothing about how logical realists might respond to it. Thus, I am left wondering why Resnik thinks logical anti-realism is superior to empiricist versions of logical realism. Perhaps his anti-realism here is based on a Quinean distaste for modality. Perhaps he thinks that the very notion of modality is just *prima facie* suspect and that, because of this, the burden of proof is on logical realists. This would explain why he spends so much time blocking arguments for realism.

In any event, Resnik's attempts to block the arguments for logical realism are interesting for those of us who think that some sort of logical realism is right—*i.e.*, for those of us who think that certain sentences really do entail certain others. For they force us to ask: What is the *argument* for this sort of logical realism? Well, one argument we might attempt here goes something like this: (i) if we have good reason to believe that '*A*' and 'if *A* then *B*' are true, then it is rational to conclude that '*B*' is true; but (ii) the only way to account for this is to claim that '*A*' and 'if *A* then *B*' *entail* '*B*'; and (iii) logical anti-realists cannot claim that '*A*' and 'if *A* then *B*' entail '*B*' because they are anti-realists about entailment.

Resnik's response to this argument is that realists are, in fact, no better off than anti-realists here, that they have merely moved the problem back a stage. More specifically, Resnik claims that just as we can ask logical anti-realists why it is rational to infer '*B*' from '*A*' and 'if *A* then *B*', so too, we can ask realists why it is rational to infer '*B*' from '*A*', 'if *A* then *B*', and '"*A*" and "if *A* then *B*" entail "*B*"'. According to Resnik, both realists and anti-realists can be driven into a Carrollian regress here, and the only way for either of them to stop the regress is to eventually say something like 'It just *is* rational to infer that "*B*" is true.'

It seems to me that logical realists can respond here by pointing out that when anti-realists ask:

> Why should we infer '*B*' from '*A*', 'if *A* then *B*', and '"*A*" and "if *A* then *B*"

entail "B"'?,

they will answer not by saying 'You just should', as Resnik suggests, but by saying:

> Wait a minute, you don't understand. We're not saying that you should infer 'B' from 'A', 'if A then B', and '"A" and "if A then B" entail "B"'. We're saying that you should infer 'B' from 'A' and 'if A then B' by themselves. For these two sentences already *entail* 'B'. That is, if 'A' and 'if A then B' are true, then 'B' *must* be true, *i.e.*, it *couldn't* be false.

Now, I actually think that Resnik can provide an analogous explanation here of why it is rational to infer 'B' from 'A' and 'if A then B'. That is, I think that he too can do better than the 'You just should' response. For he can say that it is rational to infer 'B' here and, more generally, to reason according to the principles of our inferential practices, because these practices *work*. But this, I think, leads to an argument for logical realism; for realists can explain *why* our inferential practices work, and Resnik cannot.[8]

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[8] I would like to thank Michael Resnik for commenting on an earlier draft of this review.